MA 410, Fall 2011

List of Topics for the Final.

For the Final, you need to 1) know definitions and corresponding notation, examples and statements of theorems covered in the list below and 2) be familiar with proofs to such extent that you can use similar arguments to solve problems.

Final will be Open Book: any books and any notes. Cooperation will be forbidden. Electronic means of communication (cell phones, smartphones, tablet pc, laptops, etc) will be forbidden, with the following exception. If you have an electronic version of a textbook, you can use the corresponding device for reading the textbook during Exam. In this case you will need to show me both the device and textbook.

(1) *Calculus in $\mathbb{R}^3$ in terms of tangent vectors and differential forms.* (1.1–1.7, 2.1 in O'Neill)
- Euclidean space $\mathbb{R}^3$.
- Tangent vectors to $\mathbb{R}^3$, vector fields.
- Differentiable functions $\mathbb{R}^3 \to \mathbb{R}$. Directional derivatives, basic properties of derivation: linearity, Leibniz rule.
- Curves in $\mathbb{R}^3$. Reparametrization of a curve. Velocity of a curve. Regular curves.
- 1-forms in $\mathbb{R}^3$, abstract definition (as a functional), local definition (through coordinates). Differential of a function $\mathbb{R}^3 \to \mathbb{R}$. Expression of differential through partial derivatives. Basic properties of differential (linearity, Leibniz rule, composition).
- Differential forms in $\mathbb{R}^3$, local definition. Wedge product. Exterior derivative d. Basic properties of d (linearity, Leibniz rule w.r.t. wedge product).
- Differentiable mappings $\mathbb{R}^n \to \mathbb{R}^m$ (definition in terms of partial derivatives, optionally — in terms of linear approximation). Tangent map. Jacobian, relation to the tangent map. Action of tangent map on velocity of curve. Regular mappings. Inverse function theorem (proof not required).
- Linear algebra in $\mathbb{R}^3$: Euclidean dot product, Euclidean norm of a vector, cross product in $\mathbb{R}^3$. Basic properties of dot and cross products, of norm. Shwarz inequality. Cross product as determinant. Triple product. Triple product as determinant. Frame in $\mathbb{R}^3$, attitude matrix. Relation of dot product and coordinates in arbitrary frame.

(2) *Curves in $\mathbb{R}^3$.* (Sections 2.2–2.4)
- Curves in $\mathbb{R}^3$. Velocity, speed, arc length of a curve. Arc-length parametrization. Vector field on a curve, derivative of a vector field, acceleration of a curve.
- Frenet apparatus for unit-speed curve: definitions of tangent vector T, principal normal vector N and binormal vector B, curvature κ, torsion τ. Frenet formulas. Approximation of a curve using Frenet frame (Frenet approximation). Osculating plane, osculating circle.
- Frenet apparatus for arbitrary speed curve: computing T, N, B, κ, τ. Frenet formulas for arbitrary speed curve.

— *see next page* —

(3) *Euclidean geometry of* $\mathbb{R}^3$. (Sections 3.1–3.5)

- Isometries of $\mathbb{R}^3$. Tranlations, planar rotations, orthogonal transformations. Arbitrary isometry of $\mathbb{R}^3$ as a composition of a translation and an orthogonal map. Tangent map of isometry. Frame transitivity of isometries (existence and uniqueness of an isometry whose tangent map carries given frame to other given frame).
- Orientation of a frame in $\mathbb{R}^3$. Orientation-preserving and orientation-reversing isometries. Action of tangent map of an isometry on dot product, on cross product, on triple product.
- Properties preserved by isometries. Isometry preserves derivative of a vector field. Action of isometries on Frenet frame. Congruence of curves. Unit-speed curve is determined by its curvature and torsion up to an isometry. Arbitrary-speed curve is determined by its speed, curvature and torsion up to an isometry. Criterion (in terms of curvature/torsion) for a curve to be a part of straight line, circle, circular helix.

(4) *Calculus on a Surface in* $\mathbb{R}^3$. (Sections 4.1 – 4.8)

- Definition of a surface in $\mathbb{R}^3$. Proper coordinate patches. Monge patches. Monge surfaces, level surfaces, surfaces of revolution. Parametrizations. Parametrization of surface of revolution, of ruled surface.
- Differentiable functions on a surface; valued in a surface. Description of differentiable maps $\mathbb{R}^n \to M$. Smooth overlap theorem.
- Tangent vectors to a surface in $\mathbb{R}^3$. Tangent plane, basis of a tangent plane. Euclidean vector field, tangent vector field, normal vector field. Gradient. Derivative of a function with respect to a tangent vector.
- Differential forms on a surface M in $\mathbb{R}^3$. Abstract definition of a 2-form. Abstract definition of a wedge product of two 1-forms. Abstract definition of exterior derivative d of a 1-form, consistency of the definition. Basic properties of exterior derivative (d^2, linearity, Leibniz properties). Closed and exact forms.
- Differentiable maps between surfaces. Tangent map $F*$. Pullback map F^*. Diffeomorphisms of surfaces.
- Integration of forms. Integral of a 1-form over a curve. Integral of a differential, path independent integrals. Integral of a 2-form over a segment. Stokes' theorem for integral over a 2-segment.
- Elements of topology. Connected surface (pathwise connected). Compact surface in $\mathbb{R}^3$ (either open cover definition, or finite union of 2-segments definition, or bounded closed subset definition; no proofs). Maximum-minimum theorem for compact surfaces (no proof).
- Elements of topology. Orientable surfaces in $\mathbb{R}^3$. Equivalence of two definitions.
- Elements of topology. Curves homotopic to a constant. Simply connected surfaces. Integral of a closed form along a curve homotopic to a constant. Poincaré lemma (no proof). Orientability of a compact surface in $\mathbb{R}^3$ (no proof). Orientability of a simply connected surface (no proof).

- Manifolds. Definition of an abstract surface. Projective plane as an abstract surface. Tangent vectors to an abstract surface. Definition of an n-dimensional manifold. Tangent bundle as a manifold.

(5) *Geometry of a surface in* $\mathbb{R}^3$. (Sections 5.1 – 5.4, part of 5.6)
- Shape operator: definition, examples, symmetry property.
- Shape operator as a measure of bending. Normal curvature in a direction. Sign of normal curvature. Principal directions (umbilic case, non-umbilic case). Principal directions and curvatures as eigenvectors and eigenvalues. Euler's formula.
- Gaussian and mean curvature. Sign of Gaussian curvature and quadratic approximation. Expression of Gaussian and mean curvatures through tangent vector fields. Differentiability of principal curvatures. Expression of Gaussian and mean curvatures through a coordinate patch and its derivatives and shape operator. Examples.
- Geodesics of a surface in $\mathbb{R}^3$ as curves with acceleration normal to the surface. Geodesics in simple cases (plane, sphere, normal planar section of a surface).

(6) *Riemannian geometry.* (Sections 7.1, 7.3 – 7.4, part of 7.2, 7.6.)
- Inner product. Geometric surface. Basic constructions methods (conformal change, pullback, coordinate description). Metric tensor. Riemannian manifold. Frame field on a geometric surface, dual forms, Riemann connection form ω_{12}, change of basis.
- Covariant derivative: uniqueness and existence. Connection equations. Covariant derivative formula. Covariant derivative of a vector field defined an a curve. Acceleration.
- Geodesics as curves with zero acceleration. Geodesic as a solution of a differential equation. Existence and uniqueness of geodesics.
- Measuring distances and areas in a geometric surface. Area form. (Section 5.4, 6.7, lectures)
- Structural equations on a surface (no proof). Gaussian curvature of a geometric surface (no proof).
- Euler characteristic of a surface (picture proof). Euler characteristic of an orientable compact surface (picture proof). Gauss-Bonnet theorem (no proof).